Nehari Functions and Rates of Growth of the Poincaré Density

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Abstract

Let N be the class of functions in the unit disc D satisfying Nehari's sufficient condition for univalence, $(1-|z|^2)^2|Sf(z)| \leq 2$. Here Sf is the Schwarzian derivative. Let λ denote the Poincaré density in the image $f(D)$. Motivated by recent results that show that, aside from an exceptional case, lower bounds for $|\nabla \log \lambda|$ can be comparable to as big as λ and as small as $\sqrt{\lambda}$, we study in this paper the lower bounds by the intermediate powers of the Poincaré density. We establish the corresponding optimal rates of growth of $(1-|z|^2)^2|Sf(z)|$ as $|z| \to 1$.

1. Introduction

Let f be locally univalent in the unit disc D. The Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$ plays a central role in characterizing the global injectivity of f, as discovered originally by Nehari. In 1949 he proved that

$$|Sf(z)| \leq \frac{6}{(1-|z|^2)^2}$$

was necessary for the univalence of f in D, while

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2} \tag{1.1}$$

was sufficient ([9]). Both constants 2 and 6 were shown also to be sharp. Two very useful aspects of the Schwarzian are its composition formula

$$S(g \circ f) = ((Sg) \circ f)(f')^2 + Sf\,,$$

and the fact that $Sg = 0$ identically if and only if $g = T$ is a Möbius transformation. As a result one has that $S(T \circ f) = Sf$ for T Möbius, and that the quantity

$$\sup_{|z|<1} (1-|z|^2)^2|Sf(z)|$$

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is unchanged under composition of f with automorphisms of D.

The class of (univalent) functions satisfying (1.1) will be denoted by N, and its members will be called *Nehari functions*. Since compositions with Möbius transformations of the image are allowed, functions in N will in general be meromorphic. Gehring and Pommerenke showed in [8] an important result concerning the structure of N: every Nehari function admits a (spherically) continuous extension to $\overline{D}$, and the image $f(D)$ fails to be Jordan if and only if

$$f = T \circ L \circ \sigma\,,$$

where T, σ are Möbius, $\sigma(D) = D$, and

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}\,.$$

The function L maps D onto a parallel strip and has

$$SL(z) = \frac{2}{(1-z^2)^2}\,.$$

In a series of recent studies of the class N, significant information has been obtained concerning geometric and analytic properties of Nehari domains $f(D)$ and their Poincaré density ([2],[3],[6],[7]). Recall that for any function f univalent in D, at a point $w = f(z)$ in the image the Poincaré density is defined by the equation

$$\lambda(w)|f'(z)| = \frac{1}{1-|z|^2}\,. \tag{1.2}$$

An important characterization of the class N comes in terms of a property of convexity of the function $\lambda = \lambda(w)$. Let f be univalent in D, and let $\Omega = f(D)$. We say that a real-valued function h defined in Ω is *hyperbolically convex* if it is convex relative to the metric $\lambda(w)|dw|$. This means that for every arclength parametrized hyperbolic geodesic $\gamma = \gamma(t)$ in Ω, the function $h(\gamma(t))$ is convex in the usual sense. As it turns out, (1.1) implies that $\sqrt{\lambda}$ is hyperbolically convex. Since (1.1) is invariant under Möbius changes $T \circ f$, the convexity of $\sqrt{\lambda}$ will be true for the Poincaré density of every Möbius shift of the image $f(D)$. On the other hand, only the Möbius invariant (hyperbolic) convexity of $\sqrt{\lambda}$ implies that $f \in N$ ([6]). Note that for the function L, there is an entire geodesic in the image (the real line) along which λ is constant. The constant value corresponds also to the absolute minimum of λ in the strip, and hence $\nabla\lambda$ vanishes identically there. In fact, one can show that if $f \in N$ is such that λ has more than one critical point, then, up to automorphisms of D, $f = aL + b$ ([3]). This is, in some sense, an analytic analogue of the result of Gehring and Pommerenke, the difference being that the Poincaré density of all bounded Nehari domains has exactly one critical point, including, in particular, the Möbius shifts $T(L(D))$ that are bounded.

Let $f \in N$, and suppose that $f(0) = 0, f'(0) = 1$. In much of our previous work we have made use of a crucial normalization, namely that $f''(0) = 0$. This can be achieved via the change

$$g = \frac{f}{1+af}\,,$$

where $2a = f''(0)$. A function satisfying $f(0) = 0, f'(0) = 1, f''(0) = 0$ will be called *normalized*, with the corresponding class being denoted by N_0. For example, unless f is a rotation of L, a normalized Nehari function will be bounded ([3]). The normalization produces a critical point of λ at $0 = f(0)$ in the image Ω, and relevant geometric information is obtained by studying the degree of convexity that $\sqrt{\lambda}$ will exhibit along geodesic rays emanating from the origin. It follows from the paragraph above, that unless f maps onto a parallel strip, then $\sqrt{\lambda}$ will grow at least linearly in each radial direction. As was shown in [3], this implies that for some constant $a > 0$

$$|\nabla \log \lambda| \geq a|w|\sqrt{\lambda}\,,\, w \in \Omega\,. \tag{1.3}$$

The bound in the right hand side will be sharp in the exponent $1/2$ of λ along a geodesic along which the rate of growth of $\sqrt{\lambda}$ is just linear. Geometrically, this will produce an outward pointing cusp in $\partial\Omega$ at the end of the geodesic. On the other hand, the highest possible rate of growth is exponential, which, as shown in [6], yields a bound of the form

$$|\nabla \log \lambda| \geq a|w|\lambda\,,\, w \in \Omega\,. \tag{1.4}$$

Such an estimate turns out to be equivalent to Ω having no exterior cusps, i.e., to Ω being a John domain. (For a detailed discussion of this concept, see, for example, [10].) This is particularly important since, within N, the John condition implies that Ω is a quasidisc ([6]).

The purpose of the present paper is to study the intermediate rates of growth of $\sqrt{\lambda}$, in particular, the corresponding lower bounds

$$|\nabla \log \lambda| \geq a|w|\lambda^{\gamma}\,,\, w \in \Omega\,, \tag{1.5}$$

where $\frac{1}{2} < \gamma < 1$. In doing so we will establish the optimal upper bounds and rates of growth near ∂D of the quantity $(1-|z|^2)^2|Sf(z)|$ that imply (1.5).

2. Main Results

We will study (1.5) by constructing a *model function*, and then applying comparison techniques. Let f be univalent in D. Using (1.2) it is not difficult to see that (1.5) corresponds to

$$\left|2\bar{z} - (1-|z|^2)\frac{f''}{f'}(z)\right| \geq a|f(z)|\left[(1-|z|^2)|f'(z)|\right]^{1-\gamma}\,.$$

Since normalized Nehari functions that are not rotations of L have bounded image, it follows that $|f(z)|$ and $|z|$ are comparable. In addition, a normalized $f \in N$ will satisfy the sharp estimate

$$\left|\frac{f''}{f'}(z)\right| \leq \frac{2|z|}{1-|z|^2} = \frac{L''}{L'}(|z|)\,, \tag{2.1}$$

with equality at a single $z \neq 0$ if and only if f is a rotation of L (see [3]). With this in mind we examine the model function $F = F_{c,\beta}$ which is the solution of the following equation:

$$(1-z^2)\frac{F''}{F'}(z) = 2z - cz\left[(1-z^2)F'(z)\right]^{\beta}\,. \tag{2.2}$$

Here $0 < \beta < 1/2$ and $c > 0$. Surprisingly, F' can be solved explicitly, and one finds that

$$F'(z) = \frac{1}{(1-z^2)\left[1 - \frac{c\beta}{2}\log(1-z^2)\right]^{\frac{1}{\beta}}}\,.$$

The denominator in the right hand side will not vanish as long as $c\beta < 2/\log 2$, which, since $\beta < 1/2$, will be the case for $c < 4/\log 2$. A simple calculation gives the Schwarzian:

$$SF(z) = \frac{2}{(1-z^2)^2} - \frac{c}{(1-z^2)\left[1 - \frac{c\beta}{2}\log(1-z^2)\right]} - \frac{c^2(1-2\beta)}{2(1-z^2)^2\left[1 - \frac{c\beta}{2}\log(1-z^2)\right]^2}\,. \tag{2.3}$$

It is clear from this that for small $c > 0$ and $0 \le x < 1$

$$0 \le SF(x) \le \frac{2}{(1-x^2)^2} \, .$$

In fact, one can show the following theorem.

Theorem 1: *If $c > 0$ is small then $F = F_{c,\beta} \in N_0$.*

Proof: Let $\delta = c\beta/2$ and $b = c^2(1-2\beta)/2$. From (2.3) we have that

$$(1-z^2)^2 SF(z) = 2 - \frac{c(1-z^2)}{1-\delta\log(1-z^2)} - \frac{b}{[1-\delta\log(1-z^2)]^2} \, .$$

Let

$$u(z) = \frac{c(1-z^2)}{1-\delta\log(1-z^2)} \quad , \quad v(z) = \frac{b}{[1-\delta\log(1-z^2)]^2} \, .$$

We will show that for small c

$$|1-v(z)| \le 1 \tag{2.4}$$

and

$$\frac{(1-|z|^2)^2}{|1-z^2|^2}\,|1-u(z)| \le 1 \, , \tag{2.5}$$

which implies that $(1-|z|^2)^2|SF(z)| \le 2$, as desired. To prove (2.4), simply observe that $\rho = 1-\delta\log(1-z^2)$ lies in the half strip $\mathrm{Re}\{\rho\} \ge 1-\delta\log 2$, $|\mathrm{Im}\{\rho\}| \le \pi\delta/2$. From this it is easy to see that (2.4) will be valid for c chosen small.

The term $u(z)$ requires a closer analysis, in particular, it is necessary to consider the additional factor $(1-|z|^2)^2/|1-z^2|^2$ that comes into play when trying to estimate the hyperbolic norm of $SF(z)$. Let $\zeta = 1-z^2 = x+iy$. Then $|1-\zeta| < 1$. Suppose first that $|\zeta| > 1/2$. We claim that already $|1-u(z)| < 1$, provided c is small. Indeed, $|1-u| < 1$ if and only if $|u|^2 < 2\mathrm{Re}\{u\}$. This is equivalent to

$$c^2|\zeta|^2 < 2c\,[x(1-\delta\log|\zeta|) - \delta y \arg\{\zeta\}] \, , \tag{2.6}$$

which will hold for the values of ζ considered if c is sufficiently small. Observe that (2.6) will be valid for small c provided ζ remains in any fixed Stolz angle $|y| \le mx$. Therefore, the only remaining case is when ζ is near the origin but close to the circle $|1-\zeta| = 1$, where $|y| = \sqrt{2x-x^2}$ becomes not comparable to x for small x. It is here that we must incorporate the term $(1-|z|^2)^2/|1-z^2|^2$. Suppose $|\zeta| \le 1/2$. Then $1-|\zeta| \ge 1-x$, hence

$$\frac{(1-|z|^2)^2}{|1-z^2|^2} = \frac{(1-|1-\zeta|^2)^2}{|\zeta|^2} \le \frac{x^2}{|\zeta|^2} \, ,$$

hence, to prove (2.5), it suffices that

$$|1-u| \le \frac{|\zeta|^2}{x^2} \, .$$

This will be the case if

$$c^2|\zeta|^2 \le 2c\,[x(1-\delta\log|\zeta|) - \delta y \arg\{\zeta\}] + \frac{|\zeta|^4 - x^4}{x^4}\,|1-\delta\log\zeta|^2 \, . \tag{2.7}$$

For ζ outside some given Stolz angle, we have $|\zeta|^2 = x^2+y^2 \ge (1+m^2)x^2$, hence $(|\zeta|^4-x^4)/x^4 \ge m^4+2m^2$, from where it follows that (2.7) will indeed hold if c is chosen small enough. This finishes the proof.

Theorem 2: *Let f be normalized and suppose that*

$$|Sf(z)| \leq SF(|z|)\,. \tag{2.8}$$

Then $f \in N$ and for some $a > 0$ the Poincaré density λ of the image $\Omega = f(D)$ satisfies

$$|\nabla \log \lambda| \geq a|w|\lambda^{1-\beta}\,.$$

Proof: It is clear that $f \in N$, and we assert that

$$\left|\frac{f''}{f'}(z)\right| \leq \frac{F''}{F'}(|z|)\,.$$

This is a generalization of the inequality stated in (2.1), and we sketch the proof; complete details can be found in [3] and [6]. Let $y = f''/f'$. Then

$$y' = Sf + \frac{1}{2}y^2\ ,\ y(0) = 0\,.$$

Since $|y|$ differentiated in any radial direction is bounded above $|y'|$, then a simple comparison argument using (2.8) establishes the claim. It also follows from this that $|f'(z)| \leq F'(|z|)$.

From (1.2) we have

$$|\nabla \log \lambda| = 2|\partial_w \log \lambda| = \frac{2}{|f'(z)|}\left|\partial_z \log(1-|z|^2)|f'(z)|\right| = \lambda\left|2\bar{z} - (1-|z|^2)\frac{f''}{f'}(z)\right| \geq$$

$$\lambda\left(2|z| - (1-|z|^2)\left|\frac{f''}{f'}(z)\right|\right) \geq \lambda\left(2|z| - (1-|z|^2)\frac{F''}{F'}(|z|)\right) = \lambda\left(c|z|\left[(1-|z|^2)F'(|z|)\right]^{\beta}\right) \geq$$

$$c|z|\lambda\left[(1-|z|^2)|f'(z)|\right]^{\beta} \geq a|f(z)|\lambda^{1-\beta}\,,$$

for some $a > 0$ chosen appropriately. This finishes the proof.

From (2.3), it is interesting to observe that

$$\lim_{x\to 1}(1-x^2)^2 SF(x) = 2\,.$$

In fact, for x close to 1, the quantity $(1-x^2)^2 SF(x)$ behaves like $2 - c'[L(x)]^{-2}$, $c' > 0$. This motivates the following theorem, where we only assume an estimate for the Schwarzian near ∂D, hence it is not required to restrict the analysis to Nehari functions.

Theorem 3: *Let f be univalent in D, with $\Omega = f(D)$ bounded. Suppose that*

$$\liminf_{|z|\to 1} L(|z|)^2\left[2 - (1-|z|^2)^2|Sf(z)|\right] = c_0 > 0\,.$$

Then for any $\beta \in (0, \frac{1}{2})$ such that

$$\frac{1-2\beta}{2\beta^2} < c_0 \tag{2.9}$$

there exists a constant $a > 0$ such that near $\partial\Omega$

$$|\nabla \log \lambda| \geq a\lambda^{1-\beta}\,. \tag{2.10}$$

Proof: Without loss of generality, we may assume that $f(0)=0$. Let $0<\beta<\frac{1}{2}$ satisfy (2.9), and choose $c_1\in(\frac{1-2\beta}{2\beta^2},c_0)$. Then there exists $r_1<1$ such that for all $r_1\le|z|<1$

$$L(|z|)^2\left[2-(1-|z|^2)^2|Sf(z)|\right]\ge c_1\,,$$

that is,

$$|Sf(z)|\le\frac{2-c_1L(|z|)^{-2}}{(1-|z|^2)^2}\,.$$

Let $F=F_{c,\beta}$ with $c>0$ small. One can verify that

$$\lim_{x\to1}L(x)^2\left[2-(1-x^2)^2SF(x)\right]=\frac{1-2\beta}{2\beta^2}\,,$$

which is less that c_1. Hence, for r_2 close to 1 and all $r_2\le|z|<1$ one has

$$|Sf(z)|\le SF(|z|)\,. \tag{2.10}$$

As in the previous theorem, we would like to estimate $|f''/f'|$ in terms of F''/F', the problem being now the absence of a common initial condition. Let $r>r_2$ and consider z_0 with $|z_0|=r$ fixed. Let

$$g=\frac{f}{1+bf}\,, \tag{2.11}$$

with b to be determined. Then

$$\frac{g''}{g'}=\frac{f''}{f'}-\frac{2bf'}{1+bf}\,, \tag{2.12}$$

and we seek that

$$\frac{g''}{g'}(z_0)=0\,.$$

Using (2.12) this gives

$$b=\frac{f''}{2(f')^2-ff''}\,,$$

evaluated at z_0. It is easy to see that from some $r>r_2$ the quantity $2(f')^2-ff''$ will not vanish on $|z|=r$, which we now fix.

Since $|Sg(z)|=|Sf(z)|\le SF(|z|)$ for $|z|\ge r$, and using the initial conditions $(g''/g')(z_0)=0,(F''/F')(r)>0$, we conclude from the comparison argument presented in the proof of Theorem 2, that

$$\left|\frac{g''}{g'}(z)\right|\le\frac{F''}{F'}(|z|)\,,$$

for all z in the radial segment $[z_0,z_0/|z_0|)$. This implies that for all such z,

$$|g'(z)|\le d_1F'(|z|)\,,$$

hence

$$|g(z)|\le d_1F(|z|)+d_2\,. \tag{2.14}$$

The constants d_1,d_2 depend on the values of g and g' at z_0. But equations (2.12) and (2.13) imply that the term $1+b(z_0)f(z_0)$ remains bounded away from zero as z_0 varies on $|z|=r$, which implies that the constants above can be chosen uniformly bounded for all such z_0.

With this we now have:

$$\left|2\bar{z}-(1-|z|^2)\frac{f''}{f'}(z)\right| = \left|2\bar{z}-(1-|z|^2)\frac{g''}{g'}(z)-\frac{2b(1-|z|^2)f'(z)}{1+bf}\right| \geq$$

$$\left|2\bar{z}-(1-|z|^2)\frac{g''}{g'}(z)\right| - \left|\frac{2b(1-|z|^2)f'(z)}{1+bf}\right| \geq 2|z|-(1-|z|^2)\left|\frac{g''}{g'}(z)\right| - \frac{2|b|(1-|z|^2)|f'(z)|}{|1+bf|} \geq$$

$$2|z|-(1-|z|^2)\frac{F''}{F'}(|z|) - \frac{2|b|(1-|z|^2)|f'(z)|}{|1+bf|} = c|z|\left[(1-|z|^2)F'(|z|)\right]^{\beta} - \frac{2|b|(1-|z|^2)|f'(z)|}{|1+bf|} \geq$$

$$c_3|z|\left[(1-|z|)^2|g'(z)|\right]^{\beta} - \frac{2|b|(1-|z|^2)|f'(z)|}{|1+bf|} \geq c_3|z|\frac{[(1-|z|)^2|f'(z)|]^{\beta}}{|1+bf|^2} - \frac{2|b|(1-|z|^2)|f'(z)|}{|1+bf|}\,.$$

If λ denotes the Poincaré density in $\Omega = f(D)$, then this chain of inequalities implies that, for w near $\partial\Omega$

$$|\nabla \log \lambda| \geq a'\frac{\lambda^{1-\beta}}{|1+bf|^2} - \frac{2|b|}{|1+bf|}\,.$$

Since Ω is bounded, $\lambda \to \infty$ near the boundary, and since $|1+bf|$ is bounded and remains bounded away from 0, the theorem follows. This finishes the proof.

We remark that without the assumption of a bounded image, (2.10) may not hold. For example, let f be as in Theorem 3 and let $w_0 \in \partial\Omega$. Let

$$g = \frac{f}{1+bf}\,,$$

where $b = -1/w_0$. Hence g is unbounded. Using that λ_f satisfies (2.10) it is not difficult to see that near the boundary of the image $g(D)$ one will have

$$|\nabla \log \lambda_g| \geq a'|1+bf|^2\lambda_f^{1-\beta} = a'|1+bf|^{2\beta}\lambda_g^{1-\beta}\,.$$

This is a weaker estimate than (2.10) because $1+bf \to 0$ as $f \to w_0$.

The following is a corollary for functions in the Nehari class.

Corollary 4: *Let $f \in N$ be normalized. If for some $c_0 > 0$*

$$|Sf(z)| \leq \frac{2-c_0|z|^2L(|z|)^{-2}}{(1-|z|^2)^2}\,, \tag{2.15}$$

then for any $\beta \in (0, \frac{1}{2})$ such that $\dfrac{1-2\beta}{2\beta^2} < c_0$ there exists a constant $a > 0$ such that

$$|\nabla \log \lambda| \geq a|w|\lambda^{1-\beta}\,,\ w \in \Omega\,. \tag{2.16}$$

Proof: Since $f \in N$ is normalized, then (2.15) implies that the image $f(D)$ is bounded. Hence the estimate claimed in (2.16) near the boundary is immediate from the previous theorem. But the Poincaré density of a bounded Nehari domain has only one critical point, which in this case occurs at 0 by the normalization. From this, (2.16) for suitable a must hold everywhere in Ω.

As mentioned in the Introduction, Nehari domains that do not satisfy (1.4) cannot be John domains. In particular, whenever (2.16) is sharp in the exponent $1-\beta$, then Ω must develop exterior cusps. It is also worth mentioning that the comparison principle underlying Theorems 1 and 2 is applicable to other similar bounds for $|Sf(z)|$ in D or near ∂D. For example, let f be univalent in D, with $f(D)$ bounded. Suppose that

$$\limsup_{|z|\to 1}(1-|z|^2)^2|Sf(z)| < 2\,. \tag{2.17}$$

It is known from [1] that $f(D)$ is a quasidisc. In order to estimate $|\nabla \log \lambda|$ we introduce the functions

$$f_\alpha(z) = \frac{1}{\alpha}\frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha}\,, \tag{2.18}$$

where $\alpha \in (0,1)$. These functions are the normalized extremals for the Ahlfors-Weill condition

$$|Sf(z)| \le \frac{2t}{(1-|z|^2)^2}\,,$$

where $\alpha = \sqrt{1-t}$, and satisfy

$$Sf_\alpha(z) = \frac{2t}{(1-z^2)^2}\,.$$

After some algebra, one can verify that

$$(1-z^2)\frac{f_\alpha''}{f_\alpha'}(z) = 2z - 2\alpha^2 f_\alpha(z)\,.$$

Since for $x>0$, $f_\alpha(x) \ge x$ it follows that

$$2x - (1-x^2)\frac{f_\alpha''}{f_\alpha'}(x) \ge \alpha^2 x\,.$$

This inequality should be viewed as the analogue of (2.2). The proof presented in Theorem 3 gives now that if f satisfies (2.17), then for some $a>0$ and near $\partial f(D)$

$$|\nabla \log \lambda| \ge a\lambda\,,$$

(see the chain of inequalities following (2.14)).

Theorem 2 (and the corresponding versions of Theorem 3) can also be generalized by considering suitable variants of (2.2). Let $\phi : [0,1] \to [0,1]$ be positive and increasing with

$$\frac{\phi'}{\phi}(s) \le \frac{1}{2s}\,, \tag{2.19}$$

and let $F = F(x)$ be the solution in $[0,1)$ of

$$(1-x^2)\frac{F''}{F'}(x) = 2x - x\phi((1-x^2)F'(x))\,,\, F'(0) = 1\,. \tag{2.20}$$

Then

$$0 \le SF(x) \le \frac{2}{(1-x^2)^2}\,, \tag{2.21}$$

and if f normalized with

$$|Sf(z)| \le SF(|z|)$$

then there exits $a > 0$ such that

$$|\nabla \log \lambda| \ge a|w|\lambda\phi(\lambda^{-1})\,.$$

Although it is clear how, in light of (2.16) and (2.18), the proof of Theorem 2 applies to give (2.19), we mention a few words concerning the details. First of all, since

$$0 \le \frac{F''}{F'}(x) \le \frac{2x}{1-x^2}$$

it follows that

$$1 \le F'(x) \le \frac{1}{1-x^2}\,,$$

therefore it is justified to evaluate ϕ at $(1-x^2)F'(x)$ in (2.20). Let

$$\mu(x) = (1-x^2)F'(x)\,.$$

Then (2.20) can be written as

$$x\phi(\mu) = -(1-x^2)\,[\log \mu]' = -(1-x^2)\frac{\mu'}{\mu}\,,$$

that is,

$$\frac{\mu'}{\mu\phi(\mu)} = -\frac{x}{1-x^2}\,.$$

In principle, this can be integrated by finding a primitive of $1/(\mu\phi(\mu))$, and then its inverse. In any case, it can be shown that

$$(1-x^2)^2 SF(x) = 2 - (1-x^2)\phi(\mu) + x^2\mu\phi(\mu)\phi'(\mu) - \frac{x^2}{2}\phi(\mu)^2\,.$$

Because ϕ takes values in $[0,1)$, it follows from (2.19) that (2.21) must hold.

3. Examples

The purpose of this section is to show that the estimate (2.15) is essentially best possible when seeking a lower bound of the form (2.16) with an exponent $1-\beta > \frac{1}{2}$. For any $\alpha > 0$ we will construct a Nehari function f such that

$$2 - (1-x^2)^2 Sf(x) \sim L(x)^{-2-\alpha}\,,\; x \to 1\,,$$

for which the Poincaré density only satisfies an estimate of the form (1.3).

Let $\Omega_0 = L(D)$, that is,

$$\Omega_0 = \{\zeta\,:\,|\mathrm{Im}\,\zeta| < \frac{\pi}{4}\,\}\,.$$

Let $\alpha > 0$. We consider f of the form $g \circ L$, where the function g defined in Ω_0 satisfies

$$Sg(\zeta) = -\frac{2\delta}{(a+\zeta^2)^{1+\alpha}}\,.$$

The function g can be expressed in the form

$$g(\zeta) = \int_0^\zeta u^{-2}(s)ds\,,$$

where

$$u'' - \frac{\delta u}{(a+\zeta^2)^{1+\alpha}} = 0\,. \tag{3.1}$$

Here a is a positive constant large enough so that $a+\zeta^2$ does not vanish in Ω_0, and $\delta > 0$ small will be chosen later. If we take $u(0) = 1, u'(0) = 0$, then f will be normalized and u will be positive and convex increasing for $y > 0$. The Schwarzian of f is given by

$$Sf(z) = Sg(\zeta)L'(z)^2 + SL(z) = \frac{2}{(1-z^2)^2}\left(1 - \frac{1}{(a+\zeta^2)^{1+\alpha}}\right)\,, \zeta = L(z)\,,$$

and one can show that for a large enough

$$|1 - \frac{1}{(a+\zeta^2)^{1+\alpha}}| < 1\,,$$

hence $f \in N$. In particular, no solution of u of the linear equation (3.1) can vanish more than once in the strip. Also, since f is normalized, the image $f(D)$ is bounded, showing that the solution u with the chosen initial conditions at 0 cannot vanish in Ω_0.

Let λ be the Poincaré density in $f(D)$. We claim that along the geodesic ray $f([0,1))$ in the image, the best lower bound for $|\nabla \log \lambda|$ is of the form (1.3). Because f is an isometry when considering the hyperbolic metrics in D and $f(D)$, it suffices to study the rate of growth of the function

$$h(z) = \frac{1}{\sqrt{(1-|z|^2)|f'(z)|}}\,,$$

relative to the hyperbolic metric in D. Since $L'(z) = \dfrac{1}{1-z^2}$ it follows that

$$h(x) = \frac{1}{\sqrt{|g'(L(x))|}} = u(L(x))\,,$$

hence

$$(1-x^2)h'(x) = u'(y)\,, \tag{3.2}$$

where $y = L(x)$. The left hand side represents the derivative of h in the x-direction relative to the hyperbolic metric in D. Via a comparison argument we will show that $u'(y)$ is bounded.

Let $v = v(y)$ be defined by

$$v(y) = 1 + a^{-\alpha}y - \int_0^y \frac{ds}{(a+s^2)^\alpha}\,.$$

Observe that $v(y)$ will not vanish for $y > 0$ and that $h(y) \sim y$ as $y \to \infty$. We have

$$v''(y) = \frac{2\alpha y}{(a+y^2)^{1+\alpha}}\,,$$

hence for δ sufficiently small and $y \geq y_0 > 0$

$$\frac{\delta}{(a+y^2)^{1+\alpha}} \leq \frac{v''}{v}(y)\,.$$

By taking δ again small enough we can arrange that $(u'/u)(y_0) \leq (v'/v)(y_0)$, from where the standard Sturm comparison theorem ensures that, for $y \geq y_0$

$$\frac{u'}{u} \leq \frac{v'}{v},$$

and $u \leq c_1 v$. Thus

$$u' \leq \frac{v}{u} v' \leq v' = c_1 \left(a^{-\alpha} - \frac{1}{(a+y^2)^\alpha} \right) \leq c_1 a^{-\alpha},$$

which, from (3.2), implies that for $x \geq x_0 = L^{-1}(y_0)$

$$(1-x^2)|\partial_z h(x)|$$

is bounded above. From this, and using that taking ∂_z of h in D corresponds to taking $|f'|\partial_w$ of $\sqrt{\lambda}$ in the image, it follows that near the end of the image $f([0,1))$

$$|\nabla \log \lambda| \sim \sqrt{\lambda}.$$

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